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A CONTINUUM MODEL FOR FIBER REINFORCED MATERIALS WITH DEBONDING

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Abstract—A continuum theory for a fiber-reinforced material with debonding between the constituents is presented. The debonding phenomenon is simulated by imposing the continuity of the normal displacements at the fiber-matrix interfaces while allowing free tangential slip there. The derived theory is of the lowest order and is obtained by using a first order expansion in the displacements in the fiber and matrix phases. The theory is applied to investigate the effect of debonding on the propagation of waves in a boron/epoxy fiber reinforced material. It is shown that an additional mode of propagation is obtained as compared with the usual case of perfect bonding.

INTRODUCTION

The majority of the research carried out on the effective behavior of fiber-reinforced composites has been based on the assumption of perfect bonding between the constituents. A comprehensive introduction to the subject of the overall behavior of composites can be found in the book by Christensen [1]. The debonding phenomenon is however well known to exist between fiber and matrix and it is important therefore to study its effect on the mechanical behavior of the composite. Due to the nonuniform occurence of the loss of bonding between fiber and matrix, the incorporation of such debonding phenomena in a continuum model is, no doubt, a tremendously difficult task.

Several attempts appear in the literature which approach this difficult and important problem on the basis of a finite-element solution. Some of the existing simplified treatments are based on allowing the loss of contact between fiber and matrix in a certain described manner and employing a numerical procedure to study the debonding phenomena [2]. Other treatments involve the introduction of a layer between the constituents in order to simulate the conditions at the fiber matrix interface [3]. In [4], a flexible bond model allowing tangential slip has been adopted between fiber and matrix, and a homogeneization method has been used in conjunction with a finite element solution to investigate the effect of debonding.

The effective behavior of perfectly bonded fiber reinforced media has been investigated by Aboudi in [5] through the use of a Legendre expansion formalism in the representative cell. The predicted equivalent moduli were examined in [5, 6] by extensive comparisons with various theoretical, numerical and experimental approaches and excellent agreement was obtained. The advantage of the method relies on the fact that it is readily extendable to the inelastic region, so that the elastoplastic effective behaviour of the composite can be determined.

In the present paper the method of [5] is modified to allow the loss of bonding between the constituents. A continuum model of the fiber reinforced composite is derived in which a simplified description of the debonding phenomenon is incorporated. The model consists of constitutive laws and equations of motion which govern the average field variables of the constituents.

In the first section, a description of the geometry of the fiber reinforced composite and the interface conditions is given. The debonding phenomena is simulated by demanding the continuity of the normal displacements at the fiber-matrix interfaces while allowing free tangential slip there. This model of complete slip in the tangential direction is certainly an extreme description of the actual debonding phenomenon which is nonuniform as well as partial. It is however expected that the model will give some information on the phenomena in shear failure.

This model of complete tangential slip was used by Drumheller in [7] to simulate debonding in a periodically bilaminated composite. The investigation was directed to study the effect of a debonding on the propagation of waves in the laminated medium, and was based on the direct use of three-dimensional elastodynamic equations. For the case of very long wavelengths compared to the thickness of the layering, it was shown in [7] that an additional mode of propagation is introduced due to debonding. The obtained speeds of propagation match satisfactorily the observed experimental results, thus justifying the use of the model in spite of its simplicity. The reported experiments in [7] involve impacting a bilaminated composite, thus generating a propagating stress wave through the specimen. The observed additional speed of propagation was explained in [7] through the theoretical predictions based on the debonding model. It should be noted here that such a three-dimensional approach in a fiber reinforced composite is of tremendous difficulty and does not seem to be feasible.

In the second section of the paper, the described debonding model is used in conjunction with a first order expansion in the displacements as in [5] in order to derive the continuum theory. Interface conditions between the constituents are implemented on an average basis, constitutive equations are developed and equations of motion are derived.

In the third section of the paper the derived continuum theory with debonding is applied to study the propagation of steady waves in the fiber reinforced composite. By properly selecting the dimensions of the fibers in a limiting situation, a periodically bilaminated medium with debonding is obtained. The predicted wave speeds in this special case as based on the derived theory, are in excellent agreement with those given by Drumheller in [7]. This provides a check of the present theory in a situation for which a three-dimensional solution exists.

Results are given for a fiber reinforced material with boron and epoxy constituents. The speeds of steady waves propagating at any direction with the fibers are given for various chosen values of reinforcements ratios. These wave speeds are compared with those which arise in the corresponding situation of a perfectly bonded fiber reinforced composite. It is shown that an additional mode of propagation is introduced by the presence of the debonding phenomenon.

In order to give better insight into the used debonding model and the obtained propagation phenomena, the paper is terminated by an appendix that considers a more general debonding model which includes the present one as a special case. An exact treatment is given in the framework of three-dimensional elasticity for the simplest case of SH waves propagating in the direction of the layering of a periodically bi-laminated medium with debonding. A flexible bond model is used which states that the jump in the tangential displacements at the constituent interfaces is proportional to the shear traction there. Limiting values of the flexibility parameter yields perfect bonding on one hand and the debonding model used in the paper on the other hand. The influence of the flexible bond to the propagation of harmonic waves is studied.

1. GEOMETRY OF THE COMPOSITE AND INTERFACE CONDITIONS

Consider a composite material which consists of unidirectional elastic fibers embedded in an elastic matrix. The composite will be modeled by a doubly periodic array of rectangular fibers as shown in Fig. 1. Let d_1, h_1 denote the dimensions of the rectangular cross section of the fibers and d_2, h_2 represent the spacing of the fibers within the matrix in the x_2 and x_3 directions respectively.

A representative cell of the composite can be chosen as a rectangle with a fiber located at one of its corners. The cell NFHS in Fig. 2 for example is a representative cell. It contains four subcells whose center coordinates are denoted by $(x_2^{(\alpha)}, x_3^{(\beta)})$. Here, and in the sequel α and β will indicate that quantities belong to one of the subcells and repeated α or β will not imply summation. At the center of each subcell local coordinates are introduced and denoted by $(\bar{x}_2^{(\alpha)}, \bar{x}_3^{(\beta)})$.

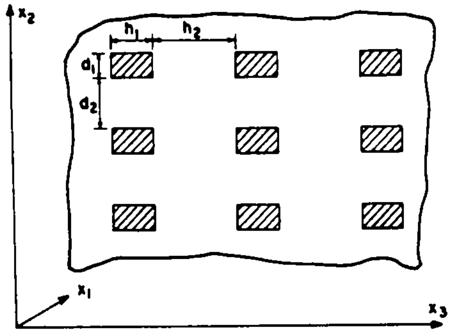

Fig. 1. A fiber-reinforced composite in which the rectangular fibers are arranged in a doubly periodic array.

There exists relations between the subcell center coordinates $(x_2^{(\alpha)}, x_3^{(\beta)})$ and the coordinates of subcell interfaces in the kth cell denoted by $(x_2^{(k)}, x_3^{(k)})$. For example for the cell NFHS and the subcell JFGK we have

$$x_3^{(1)} = x_3^{(k)} - h_1/2, \quad x_2^{(1)} = x_2^{(k)} + d_1/2. \tag{1}$$

Note that if the cell ACKI was chosen as the representative cell with subcell interface coordinates $x_2^{(k)}$, $x_3^{(k)}$ (these have not been shown in the figure for simplicity), then the relations of eqn (1) for the center of the subcell JFGK would have been

$$x_3^{(1)} = x_3^{(k)} + h_1/2, \quad x_2^{(1)} = x_2^{(k)} - d_1/2. \tag{2}$$

Let us denote the displacements and stresses in each subcell by $u_i^{(\alpha\beta)}$ and $\sigma_{ij}^{(\alpha\beta)}$ respectively with $i, j = 1, 2, 3$. The debonding between fiber and matrix is simulated in this paper by allowing tangential slip between fiber and matrix and allowing complete bonding in the direction normal to the constituent interfaces. The continuity conditions

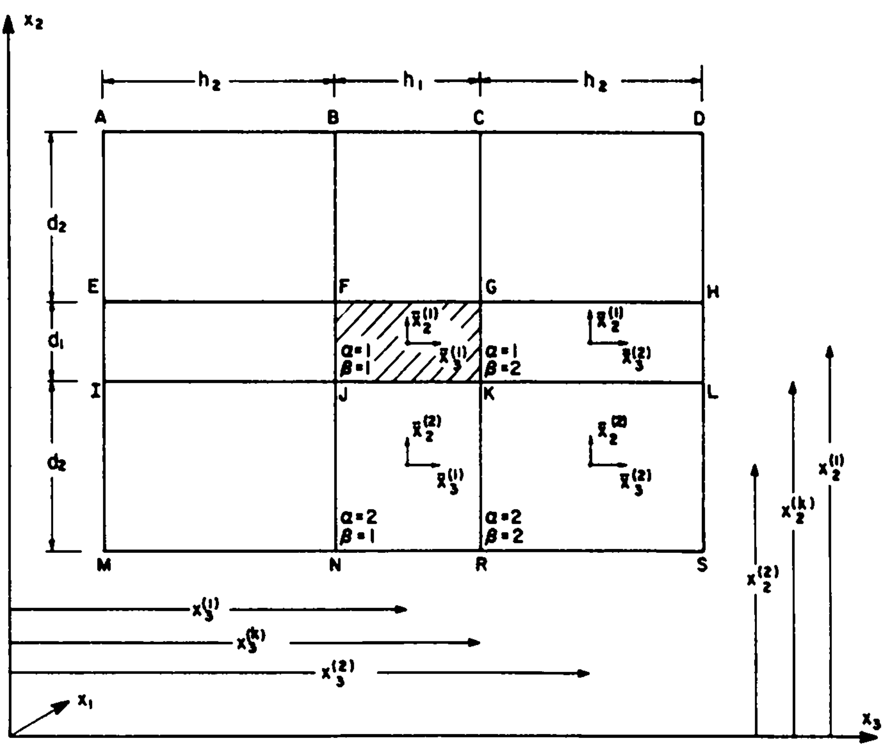

Fig. 2. Representative cells.

of the normal displacements and tractions at the fiber-matrix interfaces are therefore described by:

$$u_3^{(11)}\,|_{\bar{x}_3^{(1)}=\pm h_1/2} = u_3^{(12)}\,|_{\bar{x}_3^{(2)}=\mp h_2/2} \tag{3}$$

$$u_2^{(11)}\,|_{\bar{x}_2^{(1)}=\pm d_1/2} = u_2^{(21)}\,|_{\bar{x}_2^{(2)}=\mp d_2/2} \tag{4}$$

$$\sigma_{33}^{(11)}\,|_{\bar{x}_3^{(1)}=\pm h_1/2} = \sigma_{33}^{(12)}\,|_{\bar{x}_3^{(2)}=\mp h_2/2} \tag{5}$$

$$\sigma_{22}^{(11)}\,|_{\bar{x}_2^{(1)}=\pm d_1/2} = \sigma_{22}^{(21)}\,|_{\bar{x}_2^{(2)}=\mp d_2/2}. \tag{6}$$

The plus and minus signs in the above equations denote the two different equations obtained, depending upon whether the fiber-matrix interface follows or preceeds the cell (11).

It is noted however that due to debonding which is simulated by pure lubrication throughout, no continuity of the tangential displacements exists at the fiber-matrix interfaces and the shear tractions identically vanish there.

Between the matrix subcells (12), (22) and (22), (21) the classical continuity conditions of all displacements and tractions are enforced and they will not be reproduced here for the sake of simplicity.

2. DERIVATION OF THE CONTINUUM THEORY

A continuum model of the present fiber reinforced composite with debonding can be constructed by considering a first order expansion of the displacements in each subcell in the following form

$$u_i^{(\alpha\beta)} = W_i^{(\alpha\beta)} + \bar{x}_2^{(\alpha)}\phi_i^{(\alpha\beta)} + \bar{x}_3^{(\beta)}\psi_i^{(\alpha\beta)}. \tag{7}$$

where $W_i^{(\alpha\beta)}$ are the displacement components of the center of subcell and $\phi_i^{(\alpha\beta)}$, $\psi_i^{(\alpha\beta)}$ characterize the linear dependence of the displacements on the local coordinates within the subcell.

Expressions for the strains can be obtained by using the expansions (7) for the displacements in

$$\epsilon_{ij}^{(\alpha\beta)} = (\partial_i u_j^{(\alpha\beta)} + \partial_j u_i^{(\alpha\beta)})/2 \tag{8}$$

where $\epsilon_{ij}^{(\alpha\beta)}$ denote the strains in the subcells and the derivatives are defined by:

$$\partial_1 = \frac{\partial}{\partial x_1},\ \partial_2 = \frac{\partial}{\partial \bar{x}_2^{(\alpha)}},\ \partial_3 = \frac{\partial}{\partial \bar{x}_3^{(\alpha)}}. \tag{9}$$

Using the expansions of the strains in the usual Hooke's law for each isotropic constituent, the following expansions for the stresses $\sigma_{ij}^{(\alpha\beta)}$ are obtained.

$$\left.\begin{aligned}
\sigma_{11}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\Delta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)}\left[\frac{\partial}{\partial x_1}W_1^{(\alpha\beta)} + \bar{x}_2^{(\alpha)}\frac{\partial}{\partial x_1}\phi_1^{(\alpha\beta)} + \bar{x}_3^{(\beta)}\frac{\partial}{\partial x_1}\psi_1^{(\alpha\beta)}\right],\\
\sigma_{22}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\Delta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)}\phi_2^{(\alpha\beta)},\\
\sigma_{33}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\Delta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)}\psi_3^{(\alpha\beta)},
\end{aligned}\right\} \tag{10}$$

with

$$\left.\begin{aligned}
\Delta^{(\alpha\beta)} = &\left(\frac{\partial}{\partial x_1}W_1^{(\alpha\beta)} + \phi_2^{(\alpha\beta)} + \psi_3^{(\alpha\beta)}\right)\\
&+ \bar{x}_2^{(\alpha)}\frac{\partial}{\partial x_1}\phi_1^{(\alpha\beta)} + \bar{x}_3^{(\beta)}\frac{\partial}{\partial x_1}\psi_1^{(\alpha\beta)},
\end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned}\sigma_{12}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}\left(\frac{\partial}{\partial x_1}W_2^{(\alpha\beta)} + \phi_1^{(\alpha\beta)} + \bar{x}_2^{(\alpha)}\frac{\partial}{\partial x_1}\phi_2^{(\alpha\beta)} + \bar{x}_3^{(\beta)}\frac{\partial}{\partial x_1}\psi_2^{(\alpha\beta)}\right),\\ \sigma_{13}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}\left(\frac{\partial}{\partial x_1}W_3^{(\alpha\beta)} + \psi_1^{(\alpha\beta)} + \bar{x}_2^{(\alpha)}\frac{\partial}{\partial x_1}\phi_3^{(\alpha\beta)} + \bar{x}_3^{(\beta)}\frac{\partial}{\partial x_1}\psi_3^{(\alpha\beta)}\right),\\ \sigma_{23}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}(\psi_2^{(\alpha\beta)} + \phi_3^{(\alpha\beta)}).\end{aligned}\right\}\tag{12}$$

where $\lambda^{(\alpha\beta)}$ and $\mu^{(\alpha\beta)}$ are the Lamé constants of the material within the subcell $(\alpha\beta)$.

2(a). *Implementation of the continuity conditions*

A lowest order continuum model of the fiber reinforced composite can be constructed by employing the above expansions for the displacements and stresses and demanding the fulfillment of the continuity conditions of displacements and tractions on an average basis at the subcell interfaces.

Let us consider for example the continuity conditions of the u_3-displacements described in eqn (3). As far as the expression on the left side of the eqn is concerned, we note that for $u_3^{(11)}|_{\bar{x}_3^{(1)}=(+h_1/2)}$ the relevant interface is GK, it falls within the cells NFHS and thus (1) holds. Using the expansions for $u_3^{(11)}$ in eqn (7) and expanding this expression in a first order Taylor series about the point K, we obtain

$$\begin{aligned}u_3^{(11)}|_{\bar{x}_3^{(1)}=+h_1/2} = {} & W_3^{(11)} + (d_1/2)\frac{\partial}{\partial x_2}W_3^{(11)} - (h_1/2)\frac{\partial}{\partial x_3}W_3^{(11)}\\ & + (h_1/2)\psi_3^{(11)} + \bar{x}_2^{(1)}\phi_3^{(11)},\end{aligned}\tag{13}$$

where the expression $W_3^{(11)}$ with its derivatives and $\phi_3^{(11)}$, $\psi_3^{(11)}$ are now evaluated at the point K.

Taking the average over the interface KG and denoting the average over the interface of subcell by the sign "~", we get

$$\begin{aligned}\tilde{u}_3^{(11)}|_{\bar{x}_3^{(1)}=+h_1/2} &= (1/d_1)\int_{-d_1/2}^{d_1/2} u_3^{(11)}|_{\bar{x}_3^{(1)}=(+h_1/2)}\,d\bar{x}_2^{(1)}\\ &= W_3^{(11)} + (d_1/2)\frac{\partial}{\partial x_2}W_3^{(11)} - (h_1/2)\frac{\partial}{\partial x_3}W_3^{(11)} + (h_1/2)\psi_3^{(11)}.\end{aligned}\tag{14}$$

On the other hand, for $u_3^{(11)}|_{\bar{x}_3^{(1)}=-h_1/2}$, the relevant interface is FJ, it falls within the cell ACKI, and thus eqn (2) holds.

Carrying out an average over the interface FJ similar to eqn (14), yields

$$u_3^{(11)}|_{\bar{x}_3^{(1)}=(-h_1/2)} = W_3^{(11)} - (d_1/2)\frac{\partial}{\partial x_2}W_3^{(11)} + (h_1/2)\frac{\partial}{\partial x_3}W_3^{(11)} - (h_1/2)\psi_3^{(11)}.\tag{15}$$

Let us now define the average of a quantity Ω over a subcell $(\alpha\beta)$ by $\overline{\Omega}^{(\alpha\beta)}$ as

$$\overline{\Omega}^{(\alpha\beta)} = \frac{1}{d_\alpha h_\beta}\int_{-h_\beta/2}^{h_\beta/2}\int_{-d_\alpha/2}^{d_\alpha/2}\Omega^{(\alpha\beta)}d\bar{x}_2^{(\alpha)}\,d\bar{x}_3^{(\beta)}.\tag{16}$$

It is then seen that eqns (14) and (15) with the help of (7) and (16) can be written as

$$\tilde{u}_3^{(11)}|_{\bar{x}_3^{(1)}=\pm h_1/2} = \overline{u}_3^{(11)} \pm (d_1/2)\frac{\partial}{\partial x_2}\overline{u}_3^{(11)} \mp (h_1/2)\frac{\partial}{\partial x_3}\overline{u}_3^{(11)} \pm (h_1/2)\psi_3^{(11)}.\tag{17}$$

The same procedure, when applied to the r.h.s. of eqn (3) yields

$$\tilde{u}_3^{(12)}|_{\bar{x}_3^{(2)}=\mp h_2/2} = \bar{u}_3^{(12)} \pm (d_1/2)\frac{\partial}{\partial x_2}\bar{u}_3^{(12)} \pm (h_2/2)\frac{\partial}{\partial x_3}\bar{u}_3^{(12)} \mp (h_2/2)\psi_3^{(12)}. \quad (18)$$

By adding and subtracting eqns (17) and (18), the following result is obtained

$$\bar{u}_3^{(11)} = \bar{u}_3^{(12)}, \quad (19)$$

$$\frac{\partial}{\partial x_3}\bar{u}_3^{(11)} = \frac{\partial}{\partial x_3}\bar{u}_3^{(12)} = [h_1/(h_1+h_2)]\psi_3^{(12)} + [h_2/(h_1+h_2)]\psi_3^{(11)}. \quad (20)$$

The continuity conditions (4) will result in similar equations. The totality of equations resulting by fulfilling the continuity conditions of all the displacements between the matrix subcells (12)–(22) and (22)–(21), together with those implied by eqns (3) and (4) can be summarized as follows:

$$\bar{u}_r^{(11)} = \bar{u}_r^{(12)} = \bar{u}_r^{(21)} = \bar{u}_r^{(22)} = \bar{u}_r \qquad r = 2, 3 \quad (21)$$

$$\bar{u}_1^{(12)} = \bar{u}_1^{(22)} = \bar{u}_1^{(21)} = \bar{u}_1^{(m)}. \quad (22)$$

$$\left.\begin{aligned} d_1\phi_2^{(1\beta)} + d_2\phi_2^{(2\beta)} &= (d_1+d_2)\frac{\partial}{\partial x_2}\bar{u}_2, \\ h_1\psi_3^{(\alpha 1)} + h_2\psi_3^{(\alpha 2)} &= (h_1+h_2)\frac{\partial}{\partial x_3}\bar{u}_3. \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned} d_1\phi_3^{(12)} + d_2\phi_3^{(22)} &= (d_1+d_2)\frac{\partial}{\partial x_2}\bar{u}_3, \\ h_1\psi_2^{(21)} + h_2\psi_2^{(22)} &= (h_1+h_2)\frac{\partial}{\partial x_3}\bar{u}_2, \\ d_1\phi_1^{(12)} + d_2\phi_1^{(22)} &= (d_1+d_2)\frac{\partial}{\partial x_2}\bar{u}_1^{(m)}, \\ h_1\psi_1^{(21)} + h_2\psi_1^{(22)} &= (h_1+h_2)\frac{\partial}{\partial x_3}\bar{u}_1^{(m)}. \end{aligned}\right\} \quad (24)$$

where the overall average on a cell and the average over the three matrix subcells have been introduced in the above equations and are defined by

$$\left.\begin{aligned} \bar{\Omega} &= \frac{1}{A}\sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}\bar{\Omega}^{(\alpha\beta)}, \\ A_{\alpha\beta} &= h_\alpha d_\beta, \qquad A = \sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}. \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} \bar{\Omega}^{(m)} &= (A_{12}\bar{\Omega}^{(12)} + A_{21}\bar{\Omega}^{(21)} + A_{22}\bar{\Omega}^{(22)})/A^{(m)}, \\ A^{(m)} &= A_{12} + A_{21} + A_{22}. \end{aligned}\right\} \quad (26)$$

The average on the fiber will be denoted in the sequel by

$$\bar{\Omega}^{(f)} = \bar{\Omega}^{(11)} \quad (27)$$

so that

$$\bar{u}_1^{(f)} = \bar{u}_1^{(11)}. \quad (28)$$

It is noted that eqns (21), (22) and (28) imply that the average displacements in the x_2 and x_3 directions in every subcell are equal to each other and thus to the overall average; on the other hand in the direction of the fiber two independent average displacements exist and are defined by $\bar{u}_1^{(f)}$, $\bar{u}_1^{(m)}$.

Let us now proceed to analyze the implications of fulfilling the continuity of tractions on an average basis between the subcells. A similar procedure to that employed in arriving to eqns (19) and (20), when applied to (6) and the second of (10) for example, can be shown to yield:

$$\left.\begin{aligned} \bar{\sigma}_{22}^{(11)} &= \bar{\sigma}_{22}^{(21)} \\ (d_1 + d_2)\frac{\partial}{\partial x_2}\bar{\sigma}_{22}^{(11)} &= \lambda^{(21)} d_2 \frac{\partial}{\partial x_1}\phi_1^{(21)} + \lambda^{(11)} d_1 \frac{\partial}{\partial x_1}\phi_1^{(11)}. \end{aligned}\right\} \tag{29}$$

It is noted that in obtaining eqn (29) the cells NFHS and ACKI have been used. These two equations have resulted by adding and subtracting the equations resulting from the interface conditions described in (6).

Demanding the continuity of the normal tractions on an average basis between all the subcells, the following equations are obtained

$$\sigma_{22}^{(1\alpha)} = \bar{\sigma}_{22}^{(2\alpha)}, \quad \bar{\sigma}_{33}^{(\alpha 1)} = \bar{\sigma}_{33}^{(\alpha 2)}, \tag{30}$$

together with

$$\left.\begin{aligned} (d_1 + d_2)\frac{\partial}{\partial x_2}\bar{\sigma}_{22}^{(1\alpha)} &= \lambda^{(2\alpha)} d_2 \frac{\partial}{\partial x_1}\phi_1^{(2\alpha)} + \lambda^{(1\alpha)} d_1 \frac{\partial}{\partial x_1}\phi_1^{(1\alpha)}, \\ (h_1 + h_2)\frac{\partial}{\partial x_3}\bar{\sigma}_{33}^{(\alpha 1)} &= \lambda^{(\alpha 2)} h_2 \frac{\partial}{\partial x_1}\psi_1^{(\alpha 2)} + \lambda^{(\alpha 1)} h_1 \frac{\partial}{\partial x_1}\psi_1^{(\alpha 1)}. \end{aligned}\right\} \tag{31}$$

It is noted that

$$\lambda^{(11)} = \lambda^{(f)}, \quad \lambda^{(12)} = \lambda^{(22)} = \lambda^{(21)} = \lambda^{(m)}. \tag{32}$$

As far as the continuity of the tangential tractions at the interfaces of the subcells is concerned, the procedure here is again similar to that used in arriving to relations (21)–(24) for the displacements and (30), (31) for the normal tractions. Two sets of equations are obtained; the first set is:

$$\bar{\sigma}_{31}^{(11)} = \bar{\sigma}_{31}^{(12)} = 0, \quad \bar{\sigma}_{21}^{(11)} = \bar{\sigma}_{21}^{(21)} = 0, \quad \bar{\sigma}_{32}^{(\alpha\beta)} = 0, \tag{33}$$

$$\bar{\sigma}_{31}^{(21)} = \bar{\sigma}_{31}^{(22)}, \quad \bar{\sigma}_{21}^{(12)} = \bar{\sigma}_{21}^{(22)}. \tag{34}$$

of the second set, only the equations which will be needed in the sequel will be given. These are

$$(h_1 + h_2)\frac{\partial}{\partial x_3}\bar{\sigma}_{31}^{(21)} = h_1\mu^{(m)}\frac{\partial}{\partial x_1}\psi_3^{(21)} + h_2\mu^{(m)}\frac{\partial}{\partial x_1}\psi_3^{(22)}, \tag{35}$$

$$(d_1 + d_2)\frac{\partial}{\partial x_2}\bar{\sigma}_{21}^{(22)} = d_2\mu^{(m)}\frac{\partial}{\partial x_1}\phi_2^{(22)} + d_1\mu^{(m)}\frac{\partial}{\partial x_1}\phi_2^{(12)}. \tag{36}$$

It turns out that eqns (30), (33) and (34) are necessary to derive the constitutive relations of the lowest order continuum theory, whereas eqns (31), (35) and (36) will be needed to obtain the equations of motion of the theory.

2(b). *Constitutive equations*

When continuity of all the displacements and tractions between the constituents of a composite are enforced, the constitutive continuum theory of the lowest order can be casted in the form of relations between average stress and average strains over a representative cell. The constants appearing in these equations are the so called equivalent moduli. On the other hand, in the present case in which tangential slip between the constituents is allowed it will be shown that the constitutive relations will involve the average constituent stresses and average constituent strains so that the overall average strains by themselves will not be sufficient to determine the overall stresses and vice versa.

Using the displacement expansions (7) in the definition of the strains (8) and taking averages on each subcell according to (16) provides:

$$\bar{\epsilon}_{11}^{(\alpha\beta)} = \frac{\partial}{\partial x_1} W_1^{(\alpha\beta)}, \; \bar{\epsilon}_{22}^{(\alpha\beta)} = \phi_2^{(\alpha\beta)}, \; \bar{\epsilon}_{33}^{(\alpha\beta)} = \psi_3^{(\alpha\beta)}, \tag{37}$$

$$\left.\begin{aligned} 2\bar{\epsilon}_{21}^{(\alpha\beta)} &= \phi_1^{(\alpha\beta)} + \frac{\partial}{\partial x_1} W_2^{(\alpha\beta)}, \\ 2\bar{\epsilon}_{31}^{(\alpha\beta)} &= \psi_1^{(\alpha\beta)} + \frac{\partial}{\partial x_1} W_3^{(\alpha\beta)}, \\ 2\bar{\epsilon}_{32}^{(\alpha\beta)} &= \phi_3^{(\alpha\beta)} + \psi_2^{(\alpha\beta)}. \end{aligned}\right\} \tag{38}$$

Using the first of (37) together with (7), (16), (26), (28) and (22) yields

$$\bar{\epsilon}_{11}^{(f)} = \frac{\partial}{\partial x_1} \bar{u}_1^{(f)}, \; \bar{\epsilon}_{11}^{(m)} = \frac{\partial}{\partial x_1} \bar{u}_1^{(m)}. \tag{39}$$

The average of the other normal strains over the whole cell is obtained by using the second and third of (37) together with (23) and (25). The result is:

$$\bar{\epsilon}_{rr} = \frac{\partial}{\partial x_r} \bar{u}_r \qquad r = 2, 3 \tag{40}$$

where no sum on r is implied.

Among the shear strains, the only nonvanishing ones are: $\bar{\epsilon}_{12}^{(12)}, \bar{\epsilon}_{13}^{(21)}, \bar{\epsilon}_{12}^{(22)}, \bar{\epsilon}_{13}^{(22)}$. Using the first two of (38), (24) and (26), it can be proved that:

$$\left.\begin{aligned} 2\bar{\epsilon}_{21}^{(m)} &= [(d_1 + d_2)h_2/A^{(m)}]\left[\frac{\partial}{\partial x_2} \bar{u}_1^{(m)} + \frac{\partial}{\partial x_1} \bar{u}_2\right], \\ 2\bar{\epsilon}_{31}^{(m)} &= [(h_1 + h_2)d_2/A^{(m)}]\left[\frac{\partial}{\partial x_3} \bar{u}_1^{(m)} + \frac{\partial}{\partial x_1} \bar{u}_3\right] \end{aligned}\right\} \tag{41}$$

where $A^{(m)}$ was defined in eqn (26).

Averaging now the stresses in eqns (10) and (12) according to (16) yields

$$\left.\begin{aligned} \bar{\sigma}_{11}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\theta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)} \frac{\partial}{\partial x_1} W_1^{(\alpha\beta)}, \\ \bar{\sigma}_{22}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\theta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)}\phi_2^{(\alpha\beta)}, \\ \bar{\sigma}_{33}^{(\alpha\beta)} &= \lambda^{(\alpha\beta)}\theta^{(\alpha\beta)} + 2\mu^{(\alpha\beta)}\psi_3^{(\alpha\beta)}, \end{aligned}\right\} \tag{42}$$

where

$$\theta^{(\alpha\beta)} = \frac{\partial}{\partial x_1} W_1^{(\alpha\beta)} + \phi_2^{(\alpha\beta)} + \psi_3^{(\alpha\beta)}, \tag{43}$$

and

$$\left.\begin{aligned}\bar{\sigma}_{12}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}\left[\frac{\partial}{\partial x_1} W_2^{(\alpha\beta)} + \phi_1^{(\alpha\beta)}\right],\\ \bar{\sigma}_{13}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}\left[\frac{\partial}{\partial x_1} W_3^{(\alpha\beta)} + \psi^{(\alpha\beta)}\right],\\ \bar{\sigma}_{23}^{(\alpha\beta)} &= \mu^{(\alpha\beta)}[\psi_2^{(\alpha\beta)} + \phi_3^{(\alpha\beta)}].\end{aligned}\right\} \tag{44}$$

We now proceed to derive the constitutive equations of the continuum theory which will relate the average stresses $\bar{\sigma}_{11}^{(m)}$, $\bar{\sigma}_{11}^{(f)}$, $\bar{\sigma}_{22}$, $\bar{\sigma}_{33}$, $\bar{\sigma}_{12}^{(m)}$, $\bar{\sigma}_{13}^{(m)}$ to the average strains $\bar{\epsilon}_{11}^{(m)}$, $\bar{\epsilon}_{11}^{(f)}$, $\bar{\epsilon}_{22}$, $\bar{\epsilon}_{33}$, $\bar{\epsilon}_{12}^{(m)}$ and $\bar{\epsilon}_{13}^{(m)}$. The relations between shear stresses and strains will be first given as these are obtained the easiest.

Using eqns (44), (38), (34), (33) and (26) it is easily seen that

$$\left.\begin{aligned}\bar{\sigma}_{13}^{(m)} &= 2\mu^{(m)}\bar{\epsilon}_{13}^{(m)}\\ \bar{\sigma}_{12}^{(m)} &= 2\mu^{(m)}\bar{\epsilon}_{12}^{(m)}\end{aligned}\right\} \tag{45}$$

For later use it will be noted here that since $\bar{\sigma}_{12}^{(f)} = \bar{\sigma}_{13}^{(f)} = 0$, it follows that

$$\left.\begin{aligned}\bar{\sigma}_{13} &= (A^{(m)}/A)\bar{\sigma}_{13}^{(m)} = (A^{(m)}/A)2\mu^{(m)}\bar{\epsilon}_{13}^{(m)}\\ \bar{\sigma}_{12} &= (A^{(m)}/A)\bar{\sigma}_{12}^{(m)} = (A^{(m)}/A)2\mu^{(m)}\bar{\epsilon}_{12}^{(m)}.\end{aligned}\right\} \tag{46}$$

As far as the relations between normal stresses and strains are concerned the situation is more complicated. Examining the eqns (42) and (43) and noting that

$$\frac{\partial}{\partial x_1} W_1^{(11)} = \bar{\epsilon}_{11}^{(f)},\quad \frac{\partial}{\partial x_1} W_1^{(12)} = \frac{\partial}{\partial x_1} W_1^{(22)} = \frac{\partial}{\partial x_1} W_1^{(21)} = \bar{\epsilon}_{11}^{(m)}, \tag{47}$$

it is seen that these equations involve eight microstructure variables $\phi_2^{(\alpha\beta)}$, $\psi_3^{(\alpha\beta)}$ which should be expressed in terms of the average strains $\bar{\epsilon}_{11}^{(f)}$, $\bar{\epsilon}_{11}^{(m)}$, $\bar{\epsilon}_{22}$, $\bar{\epsilon}_{33}$. The needed eight algebraic equations are given in (23) and (30). These are to be used of course in conjunction with (40), (42) and (47). Once the average stresses $\bar{\sigma}_{11}^{(\alpha\beta)}$, $\bar{\sigma}_{22}^{(\alpha\beta)}$, $\bar{\sigma}_{33}^{(\alpha\beta)}$ are obtained as a function of the average strains after eliminating the eight microstructure variables, the average $\bar{\sigma}_{11}^{(m)}$, $\bar{\sigma}_{22}$ and $\bar{\sigma}_{33}$ can be readily constructed according to (25) and (26). Noting finally that $\bar{\sigma}_{11}^{(f)} = \bar{\sigma}_{11}^{(11)}$, these strain–stress relations can be formally represented by

$$\begin{bmatrix}\bar{\sigma}_{11}^{(f)}\\ \bar{\sigma}_{11}^{(m)}\\ \bar{\sigma}_{22}\\ \bar{\sigma}_{33}\end{bmatrix} = \begin{bmatrix}P_1 & Q_1 & R_1 & T_1\\ P_2 & Q_2 & R_2 & T_2\\ P_3 & Q_3 & R_3 & T_3\\ P_4 & Q_4 & R_4 & T_4\end{bmatrix}\begin{bmatrix}\bar{\epsilon}_{11}^{(f)}\\ \bar{\epsilon}_{11}^{(m)}\\ \bar{\epsilon}_{22}\\ \bar{\epsilon}_{33}\end{bmatrix} \tag{48}$$

where P_i, Q_i, R_i, T_i, $i = 1, 4$ are functions of the cell dimensions and constituents elastic moduli. It is readily seen from (48) that the overall average strains $\bar{\epsilon}_{11}$, $\bar{\epsilon}_{22}$, $\bar{\epsilon}_{33}$ by themselves will not determine the overall average stresses $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$, $\bar{\sigma}_{33}$ as opposed to the situation existing in perfectly bonded composites.

2(c). *Equation of motion*

The equations of motion of the continuum theory will be now obtained. Let us consider the representative cell FHSN of Fig. 1.

The equations of motion at any point of a subcell are

$$\partial_i\sigma_{ij}^{(\alpha\beta)} = \rho^{(\alpha\beta)}\frac{\partial^2 u_j^{(\alpha\beta)}}{\partial t^2},\qquad i = 1, 2, 3 \tag{49}$$

where the derivatives ∂_i were defined in eqn (9), $\partial/\partial t$ stands for differentiation with respect to time and $\rho^{(\alpha\beta)}$ denote the densities of the phases.

Consider first the equation with $i = 1$ in (49). Taking the average of this equation in the subcell (11) according to (16) and noting the vanishing of the shear tractions at the fiber-matrix interface yields

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{11}^{(f)} = \rho^{(f)}\frac{\partial^2 \bar{u}_1^{(f)}}{\partial t^2} \tag{50}$$

with $\rho^{(f)} = \rho^{(11)}$. Averaging again the equation with $i = 1$ in (49) over the three subcells (12), (22), (21), multiplying the respective equations by $(A_{12}/A^{(m)})$, $(A_{22}/A^{(m)})$, $(A_{21}/A^{(m)})$, and adding yields:

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{11}^{(m)} + I + J = \rho^{(m)}\frac{\partial^2 \bar{u}_1^{m}}{\partial t^2} \tag{51}$$

where

$$A^{(m)}I = \int_{-h_2/2}^{h_2/2}\int_{-d_1/2}^{d_1/2}\frac{\partial}{\partial \bar{x}_2^{(1)}}\sigma_{12}^{(12)}\,d\bar{x}_2^{(1)}\,d\bar{x}_3^{(2)} + \int_{-h_2/2}^{h_2/2}\int_{-d_2/2}^{d_2/2}\frac{\partial}{\partial \bar{x}_2^{(2)}}\sigma_{12}^{(22)}\,d\bar{x}_2^{(2)}\,d\bar{x}_3^{(2)} \tag{52}$$

$$A^{(m)}J = \int_{-h_2/2}^{h_2/2}\int_{-d_2/2}^{d_2/2}\frac{\partial}{\partial x_3^{(2)}}\sigma_{13}^{(22)}\,d\bar{x}_2^{(2)}\,d\bar{x}_3^{(2)} + \int_{-h_1/2}^{h_1/2}\int_{-d_2/2}^{d_2/2}\frac{\partial}{\partial \bar{x}_3^{(1)}}\bar{\sigma}_{13}^{(21)}\,d\bar{x}_2^{(2)}\,d\bar{x}_3^{(1)} \tag{53}$$

and $\rho^{(m)} = \rho^{(21)} = \rho^{(12)} = \rho^{(22)}$.

It will be now shown that,

$$I = \frac{\partial}{\partial x_2}\bar{\sigma}_{12}^{(m)}, \qquad J = \frac{\partial}{\partial x_3}\bar{\sigma}_{13}^{(m)}. \tag{54}$$

The expression for I in (52) can be written as follows

$$A_m I = \int_{-h_2/2}^{h_2/2}[\sigma_{12}^{(12)}|_{x_2^{(1)}=+d_1/2} - \sigma_{12}^{(12)}|_{x_2^{(1)}=-d_1/2}]\,d\bar{x}_3^{(2)} + \int_{-h_2/2}^{h_2/2}[\sigma_{12}^{(22)}|_{x_2^{(2)}=+d_2/2} - \sigma_{12}^{(22)}|_{x_2^{(2)}=-d_2/2}]\,d\bar{x}_3^{(2)}. \tag{55}$$

Enforcing the continuity of the σ_{12} stresses at the interface KL on an average basis yields

$$A_m I = \int_{-h_2/2}^{h_2/2}[\sigma_{12}^{(12)}|_{x_2^{(1)}+(d_1/2)} - \sigma_{12}^{(22)}|_{x_2^{(2)}=-(d_2/2)}]\,d\bar{x}_3^{(2)}. \tag{56}$$

An expression for $\sigma_{12}^{(12)}|_{x_2^{(1)}=+(d_1/2)}$ is obtained by considering the expansion given in the first of (12) and expanding the quantities $\partial/\partial x_1\, W_2^{(12)}$, $\phi_1^{(12)}$, $\phi_2^{(12)}$, $\psi_2^{(12)}$ in a first order Taylor series about the point K. Noting that,

$$x_2^{(1)} = x_2^{(k)} + d_1/2, \quad x_3^{(2)} = x_3^{(k)} + h_2/2 \tag{57}$$

yields

$$\int_{-h_2/2}^{h_2/2}\sigma_{12}^{(12)}|_{x_2^{(1)}=+d_1/2}\,d\bar{x}_3^{(2)} = \left[\bar{\sigma}_{12}^{(12)} + (d_1/2)\frac{\partial}{\partial x_2}\bar{\sigma}_{12}^{(12)} + (h_2/2)\frac{\partial}{\partial x_3}\bar{\sigma}_{12}^{(12)} + (d_1/2)\mu^{(m)}\frac{\partial}{\partial x_1}\phi_2^{(12)}\right]h_2. \tag{58}$$

As far as $\sigma_{12}^{(22)}|_{x_2^{(2)}=-(d_2/2)}$ is concerned, it is noted that this time for a Taylor series about the point K,

$$x_2^{(2)} = x_2^{(k)} - d_2/2, \quad x_3^{(2)} = \bar{x}_3^{(k)} + h_2/2 \tag{59}$$

thus yielding

$$\int_{-h_2/2}^{h_2/2} \sigma_{12}^{(22)}|_{x_2^{(2)}=(-d_2/2)} \, d\bar{x}_3^{(2)} = \left[\bar{\sigma}_{12}^{(22)} - (d_2/2)\frac{\partial}{\partial x_2}\bar{\sigma}_{12}^{(22)} + (h_2/2)\frac{\partial}{\partial x_3}\sigma_{12}^{(22)} - (d_2/2)\mu^{(m)}\frac{\partial}{\partial x_1}\phi_2^{(22)} \right] h_2. \tag{60}$$

Substituting (58) and (60) in (56) and using the second of (34) provides:

$$A^{(m)}I = [(d_1 + d_2)h_2/2]\frac{\partial}{\partial x_2}\bar{\sigma}_{12}^{(12)} + h_2\mu_m \times \left[(d_1/2)\frac{\partial}{\partial x_1}\phi_2^{(12)} + (d_2/2)\frac{\partial}{\partial x_1}\phi_2^{(22)} \right]. \tag{61}$$

Using now eqn (36) in (61) in conjunction with the second of (34), and noting the second of (33) together with the definition in (26), it is easily seen that the first of (54) is obtained.

The second equality in (54) is similarly proved by applying similar steps to the integrals in (53). The governing equation for $\bar{u}_1^{(m)}$ thus becomes:

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{11}^{(m)} + \frac{\partial}{\partial x_2}\bar{\sigma}_{12}^{(m)} + \frac{\partial}{\partial x_3}\bar{\sigma}_{13}^{(m)} = \rho^{(m)}\frac{\partial^2 \bar{u}_1^{(m)}}{\partial t^2}. \tag{62}$$

The other two equations of motion for $i = 2, 3$ in (49) can be similarly proven to yield

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{12} + \frac{\partial}{\partial x_2}\bar{\sigma}_{22} = \bar{\rho}\frac{\partial^2 \bar{u}_2}{\partial t^2} \tag{63}$$

$$\frac{\partial}{\partial x_1}\bar{\sigma}_{13} + \frac{\partial}{\partial x_3}\bar{\sigma}_{33} = \bar{\rho}\frac{\partial^2 \bar{u}_3}{\partial t^2} \tag{64}$$

where $\bar{\rho}$ is the overall average density of the composite defined by

$$\bar{\rho} = (A^{(11)}\rho^{(f)} + A^{(m)}\rho^{(m)})/A. \tag{65}$$

Equations (50), (62)–(64) define the equations of motion of the derived continuum model of the composite with debonding. It is noted that in contrast to the classical equivalent modulus theory, the displacements $\bar{u}_1^{(m)}$ and $\bar{u}_1^{(f)}$ are independent of each other and four equations of motion result instead of three.

3. APPLICATION: THE PROPAGATION OF STEADY WAVES IN A COMPOSITE WITH DEBONDING

The derived continuum theory will be now applied to investigate the propagation of plane waves in the fiber reinforced composite. It should be noted that since the derived theory is of the lowest order, these waves have to be with wavelengths much larger than the dimensions of the representative cell. To obtain the four differential equations in the displacements $\bar{u}_1^{(f)}$, $\bar{u}_1^{(m)}$, $\bar{u}_2$, and $\bar{u}_3$ expressions relating the average stresses to these displacements should be substituted in the equations of motion.

The relations between the shear stresses and the displacement gradients are readily obtained by using (41), (45) and (46). These yield

$$\left.\begin{aligned} \bar\sigma_{13}^{(m)} &= \tilde V\left(\frac{\partial}{\partial x_1}\bar u_3 + \frac{\partial}{\partial x_3}\bar u_1^{(m)}\right), \quad \bar\sigma_{12}^{(m)} = \tilde W\left(\frac{\partial}{\partial x_1}\bar u_2 + \frac{\partial}{\partial x_2}\bar u_1^{(m)}\right) \\ \bar\sigma_{13} &= V\left(\frac{\partial}{\partial x_1}\bar u_3 + \frac{\partial}{\partial x_3}\bar u_1^{(m)}\right), \quad \bar\sigma_{12}^{(m)} = W\left(\frac{\partial}{\partial x_1}\bar u_2 + \frac{\partial}{\partial x_2}\bar u_1^{(m)}\right) \end{aligned}\right\} \tag{66}$$

with

$$\left.\begin{aligned} \tilde V &= \mu_m(h_1 + h_2)d_2/A^{(m)}, \qquad \tilde W = \mu_m(d_1 + d_2)h_2/A^{(m)} \\ V &= (A^{(m)}/A)\tilde V, \qquad\qquad W = (A^{(m)}/A)\tilde W. \end{aligned}\right\} \tag{67}$$

The relations between the normal stresses and the displacement gradient result from eqns (48), (39) and (40).

In this paper waves propagating in the x_1x_2 plane (see Fig. 1) will be investigated. Substitution of the stress-displacement gradient relations into the equations of motion (50), (62)–(64) and setting $\partial/\partial x_3 = 0$ yield:

$$\left.\begin{aligned} &P_1\frac{\partial^2}{\partial x_1^2}\bar u_1^{(f)} + Q_1\frac{\partial^2}{\partial x_1^2}\bar u_1^{(m)} + R_1\frac{\partial^2}{\partial x_1\partial x_2}\bar u_2 = \rho^{(f)}\frac{\partial^2\bar u_1^{(f)}}{\partial t^2}, \\ &P_2\frac{\partial^2}{\partial x_1^2}\bar u_1^{(f)} + Q_2\frac{\partial^2}{\partial x_1^2}\bar u_1^{(m)} + R_2\frac{\partial^2}{\partial x_1\partial x_2}\bar u_2 \\ &\qquad + \tilde W\frac{\partial^2}{\partial x_2^2}\bar u_1^{(m)} + \tilde W\frac{\partial^2}{\partial x_1\partial x_2}\bar u_2 = \rho^{(m)}\frac{\partial^2}{\partial t^2}\bar u_1^{(m)}, \\ &W\frac{\partial^2}{\partial x_1\partial x_2}\bar u_1^{(m)} + W\frac{\partial^2}{\partial x_1^2}\bar u_2 + P_3\frac{\partial^2}{\partial x_1\partial x_2}\bar u_1^{(f)} \\ &\qquad + Q_3\frac{\partial^2}{\partial x_1\partial x_2}\bar u_1^{(m)} + R_3\frac{\partial^2}{\partial x_2^2}\bar u_2 = \bar\rho\frac{\partial^2}{\partial t^2}\bar u_2, \\ &V\frac{\partial^2}{\partial x_1^2}\bar u_3 = \bar\rho\frac{\partial^2}{\partial t^2}\bar u_3. \end{aligned}\right\} \tag{68}$$

It is seen that the last equation in (68) is decoupled from the first three and describes the anti-plane motion of the composite. The first three equations on the other hand describe the plane motion in the x_1x_2 plane. The velocity of propagation of the anti-plane wave is $(Vn_1^2/\bar\rho)^{1/2}$.

The first three equations in (68) admit plane steady waves in the form:

$$\begin{bmatrix} \bar u_1^{(f)} \\ \bar u_1^{(m)} \\ \bar u_2 \end{bmatrix} = \begin{bmatrix} \bar U_1^{(f)} \\ \bar U_1^{(m)} \\ \bar U_2 \end{bmatrix} f(n_1x_1 + n_2x_2 - ct) \tag{69}$$

where

$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} \tag{70}$$

denotes the direction of propagation with $\mathbf{i}$ and $\mathbf{j}$ being the unit vectors in the x_1- and x_2-directions, c is the speed of propagation, and $\bar U_1^f$, $\bar U_1^{(m)}$, $\bar U_2$ denote the amplitudes of the wave. Substituting (69) in the first three of (68) yields:

$$[D]\begin{bmatrix}\bar{U}_1^{(f)}\\ \bar{U}_1^{(m)}\\ \bar{U}_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{71}$$

where $[D]$ is a three by three matrix with elements

$$\begin{aligned}
D_{11} &= P_1n_1^2 - \rho^{(f)}c^2,\quad D_{12} = Q_1n_1^2,\quad D_{13} = R_1n_1n_2\\
D_{21} &= P_2n_1^2,\quad D_{22} = Q_2n_1^2 + \bar{W}n_2^2 - \rho^{(m)}c^2,\\
D_{23} &= R_2n_1n_2 + \bar{W}n_1n_2,\quad D_{31} = P_3n_1n_2,\\
D_{32} &= Wn_1n_2 + Q_3n_1n_2,\quad D_{33} = Wn_1^2 + R_3n_2^2 - \bar{\rho}c^2.
\end{aligned} \tag{72}$$

For non-trivial solutions,

$$\det[D] = 0 \tag{73}$$

from which the three propagation speeds can be determined.

3(a). *Wave propagation in a periodically bilaminated medium*

In the special case when d_1, d_2 are kept finite while $h_1 \to \infty$ with respect to h_2, a periodically bilaminated composite is obtained in which the widths of the layers are d_1 and d_2 respectively. This case of a periodically bilaminated medium with debonding as described in the present paper has been investigated by Drumheller in [7]. His analysis is directed towards the study of the debonding effect on the propagation of waves and is based on the direct use of three-dimensional elastodynamic equations in conjunction with the assumption of long wavelengths with respect to the layering thickness $d_1 + d_2$. For the in-plane waves propagating at any direction with respect to the layering it is demonstrated in [7] that the wave speeds are governed by a cubic equation (eqn (7) in [7]). It is shown by Drumheller that an additional mode of wave propagation is introduced due to debonding and the theoretically obtained speeds of propagation match satisfactorily the experimentally observed speeds. For a periodically bilaminated composite, the method presented in this paper predicts accurately the wave speeds given in [7]. Therefore this provides a check of the present theory for a special case in which a solution based on three-dimensional elasticity can be obtained.

3(b). *Wave propagation in a fiber reinforced material*

The speeds of in-plane (x_1, x_2) waves propagating in a fiber reinforced material with the adopted debonding model are given in the framework of the present theory by the roots of eqn (73). Results will be given for square fibers ($h_1 = d_1$) arranged in the matrix in a square array ($h_2 = d_2$). The fiber volume fraction in this case is given by $v_f = [d_1/(d_1 + d_2)]^2$.

The material properties of the chosen boron-epoxy fiber reinforced material is given as follows:

	$\lambda(GPa)$	$\mu(GPa)$	$\rho(\text{gm/cm}^3)$
Boron	114.9	172.35	2.63
Epoxy	2.96	1.27	1.19

In Fig. 3, the three wave speeds c_1, c_2, c_3 normalized with respect to a non-dimensional speed defined by

$$C_m = [\mu^{(m)}(3\lambda^{(m)} + 2\mu^{(m)})(\lambda^{(m)} + \mu^{(m)})\rho^{(m)}]^{1/2} \tag{74}$$

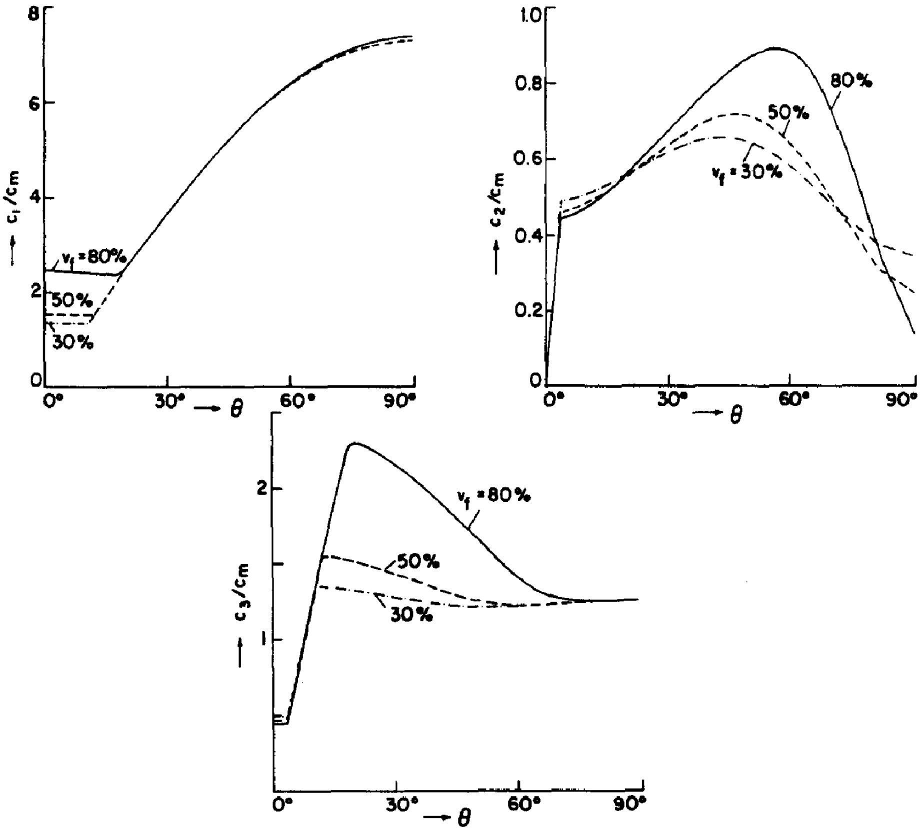

Fig. 3. The wave speeds c_1, c_2, c_3 in a fiber reinforced material with debonding, versus the propagation direction for fiber volume fractions of 30% (— — —), 50% (- - - -) and 80% (——).

are given against the direction of propagation θ. The angle θ is defined as the angle between the direction of propagation and the direction normal to the fibers, i.e. $\theta = \cos^{-1} n_2$. Results are given for three values of fiber volume fraction: $v_f = 30\%$, 50%, 80%.

It is of interest to contrast the above described curves with the corresponding ones in the case of perfect bonding. For perfectly bonded composites the lowest order con-

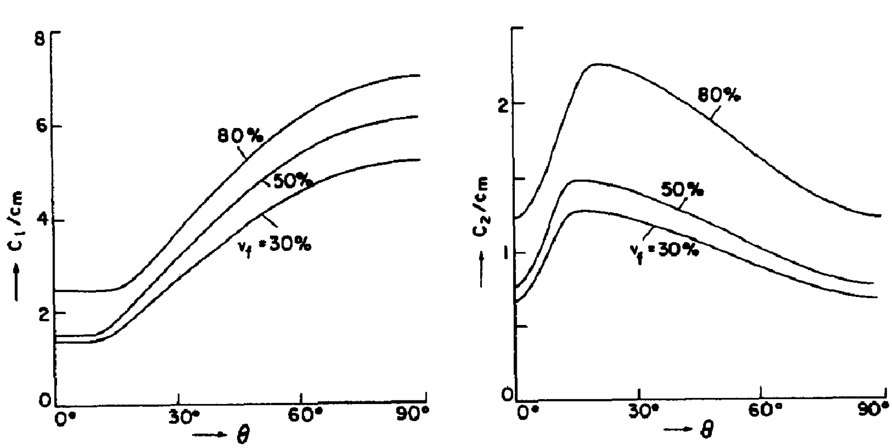

Fig. 4. The two wave speeds C_1, C_2 in perfectly bonded fiber-reinforced material versus the propagation direction for fiber volume fractions of 30, 50 and 80%.

tinuum model is given by effective modulus theory with the wave speeds being determined by the effective moduli and the average density $\bar{\rho}$. For the in-plane waves described above, there are only two wave speeds C_1, C_2 for the perfectly bonded composite. Using the equivalent moduli given in [5], these speeds are readily determined for a given value of θ. In Fig. 4, the two wavespeeds C_1, C_2 are shown against θ, for the above chosen values of the fiber reinforcement.

It is thus seen that the adopted debonding model for a fiber reinforced material predicts an additional mode of wave propagation as was noted by Drumheller in [7] in the case of a laminated medium.

For the special case in-plane waves propagating perpendicular to the fibers (i.e. $\theta = 0°$) in the composite with debonding, there are two waves with nonvanishing speeds. It can be easily shown that one of these is a dilatational wave ($\bar{u}_2 \neq 0$, $\bar{u}_1^{(f)} = \bar{u}_1^{(m)} = 0$) with a speed $c_1 = (R_3/\bar{\rho})^{1/2}$. This speed is in fact that of the dilatational wave propagating perpendicular to the fibers in a perfectly bonded composite. Thus, the dilatational mode of propagation perpendicular to the fibers is not affected by the adopted debonding mechanism.

The other wave is a shear wave ($\bar{u}_1^{(m)} \neq 0$, $\bar{u}_1^{f} = \bar{u}_2 = 0$) propagating with a speed $c_3 = (\tilde{W}/\rho^{(m)})^{1/2}$, which is certainly different than that of the shear wave propagating perpendicular to the fibers in a perfectly bonded composite.

As to in-plane waves propagating in the direction of the fibers ($\theta = 90°$), it can be readily verified that of the three existing waves, two are dilatational and the other one is a shear wave. The two dilatational waves with speeds c_1 and c_3, correspond to two different ratios of $\bar{u}_1^{f}/\bar{u}_1^{m}$, while $\bar{u}_2 = 0$. The third wave is a shear wave ($\bar{u}_2 \neq 0$, $\bar{u}_1^{(f)} = \bar{u}_1^{(m)} = 0$) with a propagation speed given by $c_2 = (W/\bar{\rho})^{1/2}$. It is noted that in the perfectly bonded composite there are two in-plane waves propagating in the fiber direction: a dilatation and a shear wave with speeds C_1 and C_2 respectively. The values of these speeds are different than those obtained in the composite with debonding.

The polarization of the waves propagating at any direction with the fibers $0° < \theta < 90°$ can be found by the standard procedure of mode determination. The obtained waves will be neither pure dilatational nor pure shear in that case.

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APPENDIX

In order to put in a better perspective the debonding model used in this paper together with the obtained propagation phenomena, we consider in this Appendix the simplest case of SH waves propagating in the direction of the layering in a doubly periodic layered media. Imperfect bonding is considered between the layering and is simulated by a flexible bond which implies that the shear traction at the interface is proportional to the jump in the tangential displacements there. The two limiting cases of perfect bonding and vanishing shear tractions are obtained as special cases of this flexible bond. The described problem, due to its simplicity, permits a treatment in the framework of exact elasticity.

Let d_α with $\alpha = 1, 2$ denote the thickness of each layer, the x_1- coordinate be in the direction of the layering and the x_2- coordinate perpendicular to the layering. Let us also define a local coordinate system at the midplane of every layer denoted by $(x_1, x_2^{(\alpha)}, x_3)$.

The anti-plane motion is defined by

$$u_3^{(\alpha)} = u_3^{(\alpha)}(x_1, x_2^{(\alpha)}, t), \qquad u_1 = u_2 = 0, \tag{A1}$$

where u_i denote the displacements and t stands for time.

The equations of motion and the constitutive equations are given by

$$\frac{\partial}{\partial x_1}\sigma_{13}^{(\alpha)} + \frac{\partial}{\partial x_2^{(\alpha)}}\sigma_{23}^{(\alpha)} = \rho^{(\alpha)}\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2} \tag{A2}$$

$$\sigma_{13}^{(\alpha)} = \mu^{(\alpha)}\frac{\partial}{\partial x_1}u_3^{(\alpha)}$$
$$\sigma_{23}^{(\alpha)} = \mu^{(\alpha)}\frac{\partial}{\partial x_2^{(\alpha)}}u_3^{(\alpha)} \tag{A3}$$

where $\mu^{(\alpha)}$ denote the shear moduli of the layers and $\rho^{(\alpha)}$ their densities.

The conditions of the interfaces are

$$\sigma_{23}^{(1)}\big|_{x_2^{(1)}=\pm d_1/2} = \sigma_{23}^{(2)}\big|_{x_2^{(2)}=\mp d_2/2} \tag{A4}$$

$$u_3^{(1)}\big|_{x_2^{(1)}=\pm d_1/2} - u_3^{(2)}\big|_{x_2^{(2)}=\mp d_2/2} = \mp R\sigma_{23}^{(1)}\big|_{x_2^{(1)}=\pm d_1/2} \tag{A5}$$

It is noted here that eqn (A5) represents the flexible bond condition with R representing the flexibility. For $R \to 0$ perfect bonding prevails and for $R \to \infty$ the shear tractions at the interfaces vanish; thus for the considered case of SH waves, the layers become independent of each other. It is important to note that in the debonding model considered in the paper, the vanishing of the shear tractions does not imply complete separation since the continuity of the normal displacements between the fibers and matrix continue to prevail.

In the framework of the exact elasticity theory, using eqns (A2), (A3) and assuming propagating waves in the form

$$u_3^{(\alpha)} = F^{(\alpha)}(x_2^{(\alpha)})\exp[ik(x_1 - ct)], \tag{A6}$$

with k being the wave number and c the phase speed, it is possible to show that

$$F^{(\alpha)}(x_2^{(\alpha)}) = A_\alpha\cos(ks_\alpha x_2^{(\alpha)}) + B_\alpha\sin(ks_\alpha x_2^{(\alpha)}) \tag{A7}$$

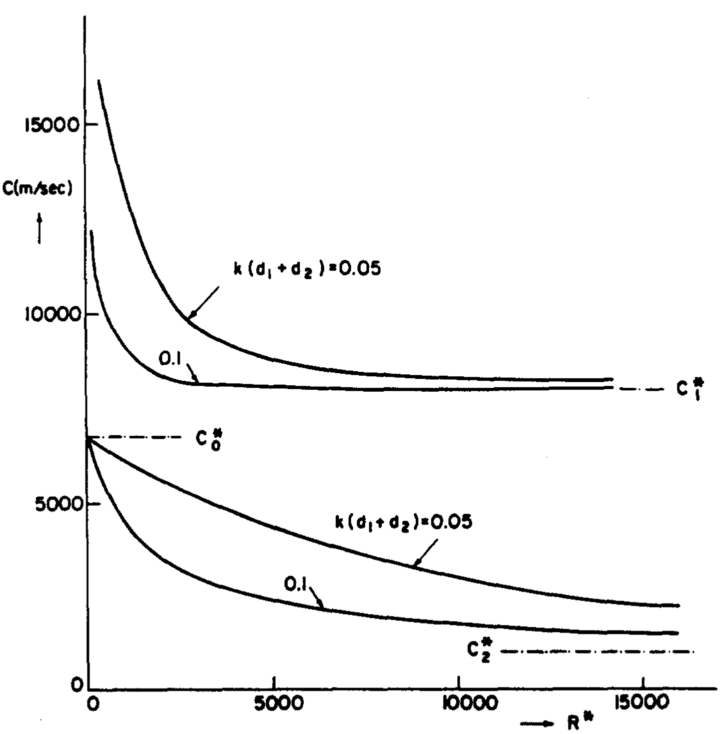

Fig. A1. The phase velocity of the first two modes for SH waves propagating in the direction of the layering in a bi-laminated medium with debonding. The phase velocity is shown versus R^* for the two first modes with the two chosen values of the wave number: $(d_1 + d_2)k = 0.1$, $(d_1 + d_2)k = 0.05$. The laminated medium is made of boron/epoxy and $d_1/(d_1 + d_2) = 0.5$.

where

$$s_\alpha = (\rho^{(\alpha)}c^2/\mu^{(\alpha)} - 1)^{1/2} \tag{A8}$$

and $A_\alpha B_\alpha$ are constants to be determined from the interface conditions (A4) and (A5).

Choosing as an illustration the symmetric motion, that is $B_\alpha = 0$, it is possible to show that (A4) and (A5) result in two homogeneous equations for A_α which furnish in their turn the following dispersion relation for the phase velocity c.

$$\gamma s_1 \tan(ks_1d_1/2) + s_2 \tan(ks_2d_2/2) = (\mu_1/\bar{\mu})k(d_1 + d_2)s_1s_2R^* \tan(ks_2d_2/2)\tan(ks_1d_1/2) \tag{A9}$$

with $\gamma = \mu_1/\mu_2$ and

$$R^* = R(d_1 + d_2)/\bar{\mu}, \qquad \bar{\mu} = (d_1\mu_1 + d_2\mu_2)/(d_1 + d_2). \tag{A10}$$

It is noted that for $R^* = 0$, the obtained relation is that given in the Appendix of [8] for the case of perfect bonding. In this situation for $k \to C$, the phase velocity is given by

$$c = c_0^* = (\bar{\mu}/\bar{\rho})^{1/2} \tag{A11}$$

where

$$\bar{\rho} = (d_1\rho_1 + d_2\rho_2)/(d_1 + d_2). \tag{A12}$$

For $R^* \to \infty$ on the other hand the dispersion relations of SH-waves propagating in two different wave-guides is obtained (see [9], p. 206). In this case for vanishing wave numbers the phase velocities become equal to

$$c_1^* = (\mu_1/\rho_1)^{1/2}, \qquad c_2^* = (\mu_2/\rho_2)^{1/2}. \tag{A13}$$

In the problem treated in this paper $R^* \to \infty$, and long wavelengths are considered; the limiting speeds however are not those of the individual phases, this being due to the fact that the normal displacements remain to be continuous at the fiber matrix interface.

For finite values of R^* the phase velocity is determined from eqn (A9) and is depicted in Fig. A1, for the two first modes of propagation. This figure shows the phase velocity versus R^* for two chosen values of the wave number k: $(d_1 + d_2)k = 0.1$ and $(d_1 + d_2)k = 0.05$. These values were chosen since the theory presented in the paper is valid for long wave-lengths compared with the micro-dimension of the composite. The graphs in this figure have been drawn for a boron/epoxy composite with a reinforcement ratio $d_1/(d_1 + d_2) = 0.5$. It is noted that for this case

$$c_0^* = 6740\ m/s, \qquad c_1^* = 8090\ m/s, \qquad c_2^* = 1030\ m/s.$$

It is seen in the figure that for $R^* = 0$ the second mode of propagation predicts infinite phase velocity; the first mode of propagation on the other hand is almost non-dispersive at vanishing R^* for the considered two values of k, and the phase velocity is practically equal to c_0^*.

For increasing values of R^* dispersion phenomena exists for both modes of propagation; this is seen by the fact that different phase velocities result for the two considered wavenumbers. For very large values of R^* on the other hand both modes become non-dispersive and the phase speeds approach the limiting wave-speeds c_1^* and c_2^* respectively.